

Decay in an asymmetric SQUID

J. P. Palomares-Báez¹ and B. Ivlev²

¹*Instituto Potosino de Investigación Científica y Tecnológica,
San Luis Potosí, San Luis Potosí 78231, Mexico*

²*Instituto de Física, Universidad Autónoma de San Luis Potosí
San Luis Potosí, San Luis Potosí 78000, Mexico*

Quantum tunneling in an asymmetric (with strongly different capacitances) SQUID is studied. Since capacitances play a role of masses one phase, related to a large mass, becomes "heavy" and remains always a constant in a tunneling process. Tunneling in an asymmetric SQUID becomes one-dimensional with a condition of optimization of tunneling probability with respect to a value of the "heavy" phase. An unusual temperature dependence of the tunneling probability is obtained. It has a finite slope at zero temperature and a transition between thermally assisted tunneling and pure activation can be not smooth depending on current through a SQUID.

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I. INTRODUCTION

A decay of a zero voltage state via quantum tunneling of a phase across a potential barrier in Josephson junctions is possible.¹⁻⁴ Tunneling in a single Josephson junction is similar to a conventional one-dimensional quantum mechanical process. In this case the tunneling mechanism is described by theory of Wentzel, Kramers, and Brillouin (WKB).⁵ Tunneling occurs from a classically allowed region which is a conventional potential well where energy levels are quantized.⁶⁻⁹ Quantum coherence between potential wells was demonstrated.¹⁰⁻¹³

Besides single Josephson junctions, superconducting quantum interference devices (SQUID) are also a matter of active investigation.¹⁴⁻²² A SQUID consists of two junctions and, therefore, represents a two-dimensional system where macroscopic quantum tunneling is also possible.

Tunneling in multi-dimensional systems was described in literature.²³⁻²⁷ According to those theories, there is a certain underbarrier path where a wave function is localized and it decays along the path. However, a tunneling scenario in two (many) dimensions can substantially differ from that main-path mechanism. For example, in a SQUID a main path can split by two ones¹⁵ and even by an infinite number of equivalent paths which interfere providing multi-path tunneling.²² See also²⁸⁻³⁰.

Capacitances of two junctions in a SQUID play a role of masses. When one capacitance is large the masses become very different and one phase becomes "heavy". This provides an additional interest for study of an asymmetric SQUID. A behavior of an asymmetric SQUID after tunneling was investigated.²¹ In this paper we also focus on strongly asymmetric SQUIDs, namely, on tunneling process in them.

During tunneling process a motion along the "heavy" coordinate is weakly generated and the process of barrier crossing becomes almost one-dimensional when the "heavy" phase is fixed. This fixed value should be determined from a condition of maximum of a tunneling probability.

That program has been performed in the paper. There are two unusual features of results.

First, the tunneling probability, as a function of temperature, has a finite slope at low temperature. This contrasts to a temperature dependence for a one-dimensional barrier where that slope is zero.

Second, a transition at a finite temperature between thermally assisted tunneling and pure activation changes its character when current approaches the critical value. At those currents temperature dependence of tunneling probability exhibits a finite jump of slopes at the transition temperature. When current is not too close to the critical value the transition is smooth as for a one dimensional barrier.

In Sec. III we apply to a SQUID a semiclassical formalism of Hamilton-Jacobi. In Sec. IV the method of classical trajectories in imaginary time is used which accounts for an optimization of tunneling probability with respect to a value of the "heavy" phase. In Sec. VII it is argued that experimental observations of the proposed phenomena in SQUID is real.

II. FORMULATION OF THE PROBLEM

We consider a dc SQUID, consisting of two Josephson junctions with phases φ_1 and φ_2 , with no dissipation when the two junctions are inductively coupled. Critical currents of the junctions are equal but capacitances, C_1 and C_2 , are different so that

$$M = \frac{C_2}{C_1} \gg 1. \quad (1)$$

A classical behavior of phases corresponds to conservation of the total energy

$$E_0 = \frac{E_J}{2\omega^2} \left[\left(\frac{\partial \varphi_1}{\partial t} \right)^2 + M \left(\frac{\partial \varphi_2}{\partial t} \right)^2 \right] + E_J \left[-\cos \varphi_1 - \cos \varphi_2 - j(\varphi_1 + \varphi_2) + \frac{1}{2\beta}(\varphi_1 - \varphi_2)^2 \right], \quad (2)$$

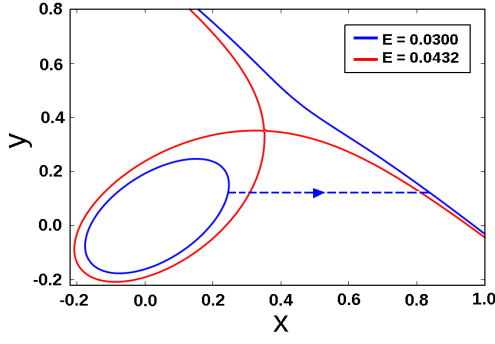


FIG. 1: (Color online) Curves of a constant energy $V(x, y) = E$ at $\alpha = 0.9$. Tunneling occurs along the dashed line, $y = y_0$, where an underbarrier wave function is localized.

where the dimensionless current $j = I/2I_c$, the Josephson energy $E_J = \hbar I_c/2e$, the plasma frequency $\omega = \sqrt{2eI_c/\hbar C_1}$, and the coupling parameter $\beta = 2\pi L I_c/\Phi_0$ are introduced. Here I_c and L are critical current and inductance of each individual junction. The magnetic flux quantum is $\Phi_0 = \pi\hbar c/e$. As follows from Eqs. (1) and (2), φ_2 is a "heavy" phase.

Below we consider a large β and the total current I close to its critical value, $(1 - j) \ll 1$. New variables are introduced by the relations

$$\begin{aligned}\varphi_1 &= \frac{\pi}{2} + (3x - 1)\sqrt{2(1 - j)} + \frac{3x}{\beta} \\ \varphi_2 &= \frac{\pi}{2} + (3y - 1)\sqrt{2(1 - j)} + \frac{3y}{\beta}.\end{aligned}\quad (3)$$

Below time is measured in the unit of

$$t_0 = \frac{\sqrt{2\beta}}{\omega} \sqrt{\frac{\alpha}{1 + \alpha}}, \quad (4)$$

where the coupling parameter is

$$\alpha = \frac{1}{\beta\sqrt{2(1 - j)}}. \quad (5)$$

The energy (2) takes the form

$$E_0 = \frac{\hbar B}{t_0} \left[\frac{1}{2} \left(\frac{\partial x}{\partial t} \right)^2 + \frac{M}{2} \left(\frac{\partial y}{\partial t} \right)^2 + V(x, y) \right], \quad (6)$$

where

$$B = \frac{B_0}{\sqrt{2}} \left(\frac{1 + \alpha}{\alpha} \right)^{5/2}, \quad B_0 = \frac{9E_J}{\hbar\omega\beta^{5/2}}. \quad (7)$$

The potential energy is

$$V(x, y) = V_0(x) + V_0(y) - \frac{2\alpha xy}{1 + \alpha}, \quad (8)$$

where $V_0(x) = x^2 - x^3$. B in Eq. (6) is called semiclassical parameter. When B is large the phase dynamics is mainly classical. Below we consider that case, $1 \ll B$.

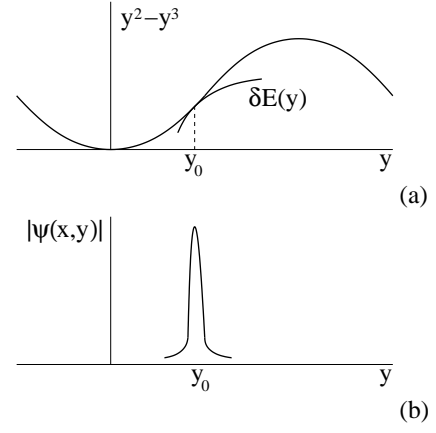


FIG. 2: (a) Form of $\delta E(y)$. (b) Corresponding density distribution is Gaussian. It is plotted for some x under the barrier.

III. DESCRIPTION OF TUNNELING

A classical dynamics of phases in a SQUID relates to Eqs. (6) and (8). The effective particle moves in the classically allowed region, in a vicinity of the point $x = y = 0$, which is restricted by the potential barrier. As known, the particle can tunnel through the barrier resulting in experimentally observable phase jumps. Character of tunneling depends on coupling strength α between the two junctions. At $\alpha = 0.90$ the curves of equal potential, $V(x, y) = E$, are shown in Fig. 1. An effective particle tunnels from one classically allowed region (the potential well) to another (the outer region).

To quantitatively study the problem of two-dimensional tunneling one should solve the Schrödinger equation with the exact potential (8). Since the potential barrier is almost classical one can apply a semiclassical method when a wave function has the form

$$\psi \sim \exp(iB\sigma), \quad (9)$$

where the classical action is $\hbar B\sigma$. σ satisfies the equation of Hamilton-Jacobi⁵

$$\frac{1}{2} \left(\frac{\partial \sigma}{\partial x} \right)^2 + \frac{1}{2M} \left(\frac{\partial \sigma}{\partial y} \right)^2 + V(x, y) = E. \quad (10)$$

We define the energy E by the relation

$$E_0 = \frac{\hbar B}{t_0}. \quad (11)$$

At a large M a solution of Eq. (10) can be written in the form $\sigma = \sigma_0 + \sigma_1$ where σ_1 is small and σ_0 is given by

$$\begin{aligned}\frac{\sigma_0(x, y)}{\sqrt{2}} &= i \int^x dx_1 \sqrt{x_1^2 - x_1^3 - \frac{2\alpha x_1 y}{1 + \alpha} - E + \delta E(x_1, y)} \\ &\quad + \sqrt{M} \int^y dy_1 \sqrt{\delta E(x, y_1) - y_1^2 + y_1^3},\end{aligned} \quad (12)$$

where $\delta E(x, y)$ is some function to be specified. It is easy to conclude that the correction σ_1 is small (proportional

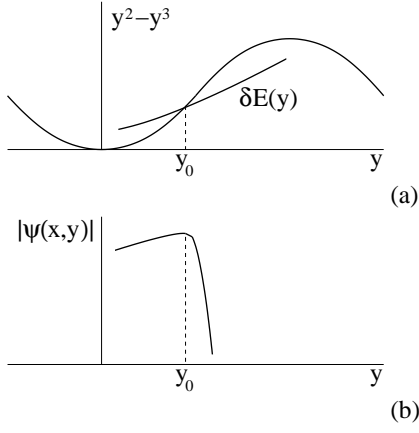


FIG. 3: (a) Form of $\delta E(y)$. (b) Corresponding density distribution, plotted for some x under the barrier, is not Gaussian. See the text.

to $1/\sqrt{M}$) when the derivative $\partial \delta E / \partial x$ is small (proportional to $1/M$). So we consider below $\delta E(y)$.

The function $\delta E(y)$ is determined by a state in the well from which tunneling occurs.

When the case of Fig. 2(a) is realized the last term in Eq. (12) provides a Gaussian distribution of density around the line $y = y_0$ shown in Fig. 2(b) for some x under the barrier. This is analogous to a conventional scenario of tunneling in two dimensions.^{25,26}

In the case of $\delta E(y)$ of Fig. 3(a) at $y_0 < y$ the density drops down under the barrier due to the second term in Eq. (12) and at $y < y_0$ due to the first one. Therefore the underbarrier density in that case is also localized in a vicinity of the line $y = y_0$. It is shown in Fig. 3(b) for some x under the barrier. This is not a Gaussian distribution but one of the type $\exp[-c(y - y_0)^{3/2}]$. At $y_0 < y$ the parameter $c \sim \sqrt{M}$ is large.

The situations in Figs. 2 and 3 relate to different types of states in the potential well from which tunneling occurs. In the case of Fig. 2 a Gaussian distribution of density holds also in the well because the last term in Eq. (12) dominates. In the case of Fig. 3 the distribution in the well is analogous to Fig. 3(b) when the part at $y < y_0$ is horizontal and, therefore, the state is distributed over a finite distance y in the well.

In the both cases, Figs. 2 and 3, tunneling occurs along the certain line $y = y_0$ and y_0 should be determined from the condition of maximum of a tunneling probability. This corresponds to classical mechanics when a particle does not move along a "heavy" direction. With an exponential accuracy a tunneling probability is the same for the both types of states in the well.

IV. TUNNELING PROBABILITY

Since tunneling occurs along the line $y = y_0$ one can use a WKB approach as in a one-dimensional case. The

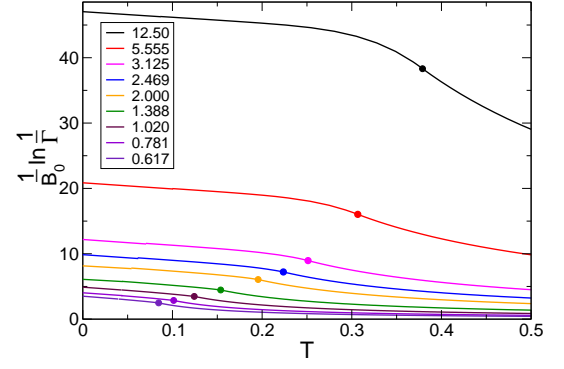


FIG. 4: (Color online) Γ is tunneling probability and temperature T is measured in the units of $\hbar\omega/\sqrt{\beta}$. The numbers mark values of the parameter $(1-j)\beta^2$.

probability of tunneling with a fixed energy E is

$$\Gamma(E) \sim \exp[-2BA(E)], \quad (13)$$

where

$$A(E) = \sqrt{2} \int dx \sqrt{v(x) - E}. \quad (14)$$

The one-dimensional potential $v(x)$ is given by

$$v(x) = x^2 - x^3 - \frac{2\alpha y_0}{1+\alpha} x + y_0^2 - y_0^3. \quad (15)$$

The integration in Eq. (14) is restricted by the classically forbidden region where $E < v(x)$.

Tunneling probability at a fixed temperature accounts for the Gibbs factor and is determined by

$$\Gamma \sim \exp \left[-2BA(E) - \frac{E_0}{T} \right], \quad (16)$$

with a subsequent optimization with respect to E . Taking Eq. (11), one can write Eq. (16) in the form

$$\Gamma \sim \exp(-2BA_T), \quad (17)$$

where

$$A_T = A(E) + \frac{E}{\theta}. \quad (18)$$

The parameter θ is connected with temperature

$$\theta = \frac{2T\sqrt{\beta}}{\hbar\omega} \sqrt{\frac{2\alpha}{1+\alpha}}. \quad (19)$$

Minimization of A_T with respect to energy defines E by the equation

$$\frac{1}{\theta} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{v(x) - E}}. \quad (20)$$

The parameter y_0 should be chosen to minimize A_T . By

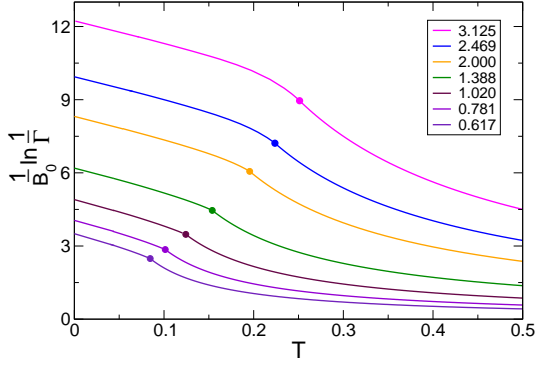


FIG. 5: (Color online) Amplification of the lower set of curves in Fig. 4. Left parts of the curves relate to a thermally assisted tunneling and right parts pertain to pure activation.

introducing imaginary time $t = i\tau$ the action A_T can be written in the form

$$A_T = \int_0^{1/\theta} d\tau \left[\frac{1}{2} \left(\frac{\partial x}{\partial \tau} \right)^2 + v(x) \right], \quad (21)$$

where the classical trajectory under the barrier is determined by Newton's equation

$$\frac{\partial^2 x}{\partial \tau^2} = 2x - 3x^2 - \frac{2\alpha y_0}{1+\alpha} \quad (22)$$

with zero velocities at the terminal points, $\tau = 0$ and $\tau = 1/\theta$. According to Eq. (20), $1/\theta$ is the underbarrier time of motion between two terminal points. In terms of trajectories, the condition of minimum A_T with respect to y_0 takes the form

$$2y_0 - 3y_0^2 = \frac{2\alpha\theta}{1+\alpha} \int_0^{1/\theta} x d\tau. \quad (23)$$

For a strongly asymmetric SQUID, a large M , tunneling occurs along a straight line $y = y_0$ shown in Fig. 1. The action (21) depends on two parameters, α and $T\sqrt{\beta}/\hbar\omega$.

The tunneling probability satisfies the relation

$$\frac{1}{B_0} \ln \frac{1}{\Gamma} = \sqrt{2} \left(\frac{1+\alpha}{\alpha} \right)^{5/2} A_T \left(\alpha, T\sqrt{\beta}/\hbar\omega \right). \quad (24)$$

A recipe of calculation of the action A_T is the following. At fixed α , y_0 , and θ one should find a solution of Eq. (22) with zero velocities, $\partial x/\partial \tau = 0$, at $\tau = 0, 1/\theta$. That solution has to be inserted into the relation (23) which defines y_0 at fixed θ and α . The solution with the defined y_0 should be substituted into Eq. (21) which produces $A_T(\alpha, T\sqrt{\beta}/\hbar\omega)$. We demonstrate in Sec. V how this scheme works in the case of low temperatures.

V. TUNNELING AT LOW TEMPERATURES

At low temperatures the energy E should be close to the minimum of the potential $v(x)$ providing a long un-

derbarrier time $1/\theta$. With the value of energy

$$E = \frac{1+2\alpha}{(1+\alpha)^2} y_0^2 \quad (25)$$

the action takes the form

$$A_T = \frac{4\sqrt{2}}{15} - \frac{2\sqrt{2}\alpha y_0}{1+\alpha} + \frac{E}{\theta}. \quad (26)$$

A minimization with respect to y_0 of the action (26), accounting for (25), is equivalent to Eq. (23). The resulting action, at low dimensionless temperature θ , is

$$A_T = \frac{4\sqrt{2}}{15} - \frac{2\alpha^2}{1+2\alpha} \theta. \quad (27)$$

VI. RESULTS

We performed a numerical solution of Eq. (22). The results for the tunneling probability are presented in Figs. 4 and 5 where temperature T is measured in the units of $\hbar\omega/\sqrt{\beta}$. Each curve in Figs. 4 and 5 consists of two parts. To the left of a dot each curve relates to above trajectory calculations corresponding to thermally assisted tunneling. To the right of a dot a curve is solely due to thermal activation

$$\frac{1}{B_0} \ln \frac{1}{\Gamma} = \frac{\hbar\omega}{T\sqrt{\beta}} \frac{2}{27\alpha^3} \begin{cases} (1-\alpha)(1+2\alpha)^2, & \alpha < 1/2 \\ 2, & 1/2 < \alpha \end{cases} \quad (28)$$

The activation energy is given by the saddle point $V(x_s, y_s)$ which coincides with the crossing point of two curves in Fig. 1. The steepest descent in Fig. 1 goes along the direction $x = y$. The saddle point $\{x_s, y_s\}$ is determined by the conditions $\partial V(x, y)/\partial x = \partial V(x, y)/\partial y = 0$. At $1/2 < \alpha$

$$x_s = y_s = \frac{2}{3(1+\alpha)} \quad (29)$$

and at $\alpha < 1/2$

$$x_s, y_s = \frac{1+2\alpha \pm \sqrt{1-4\alpha^2}}{3(1+\alpha)}. \quad (30)$$

At $\alpha < 1/2$ the transition to a pure activation regime is smooth as in a one-dimensional case. It is analogous to type II phase transition. This corresponds to $2 < (1-j)\beta^2$ in Figs. 4 and 5. At $1/2 < \alpha$ the transition to the activation regime reminds type I phase transition. A derivative with respect to temperature jumps at those points which can be observed in Fig. 5.

Numerically calculated curves in Figs. 4 and 5 match at low temperatures the analytical dependence followed from Eqs. (24) and (27). At low temperatures

$$\frac{1}{B_0} \ln \frac{1}{\Gamma} = \frac{8}{15} \left(\frac{1+\alpha}{\alpha} \right)^{5/2} \left[1 - \frac{T\sqrt{\beta}}{\hbar\omega} \frac{15\alpha^2\sqrt{\alpha}}{(1+2\alpha)\sqrt{1+\alpha}} \right]. \quad (31)$$

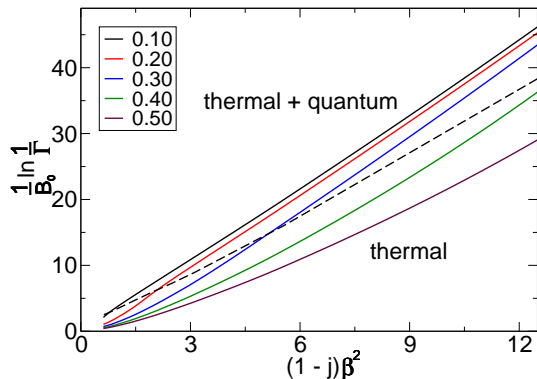


FIG. 6: (Color online) Tunneling probability Γ versus current for various values of the dimensionless temperature $T\sqrt{\beta}/\hbar\omega$. The dashed curve separates thermally assisted tunneling from a pure thermal activation.

We note that the slope in the temperature dependence (31) is finite.

The tunneling probability Γ as a function of the parameter $(1-j)\beta^2$ is plotted in Fig. 6 for different values of the dimensionless temperature $T\sqrt{\beta}/\hbar\omega$. This plot shows how Γ depends on current at a fixed temperature.

VII. DISCUSSIONS

Quantum tunneling across a one-dimensional static potential barrier is described by WKB theory. Accordingly, in two dimensions the main contribution to tunneling probability comes from an extreme path linking two classically allowed regions. The path is a classical trajectory with real coordinates which can be parametrized by imaginary time. The underbarrier trajectory is a solution of Newton's equation in imaginary time. Under the barrier the probability density reaches a maximum at each point of the trajectory along the orthogonal direction with respect to it. Along that direction the density has a Gaussian distribution. Therefore around the trajectory, which plays a role of a saddle point, quantum fluctuations are weak. The wave function, tracked along that trajectory under the barrier, exhibits an exponential decay analogous to WKB behavior. This constitutes a conventional scenario of tunneling in multi-dimensional case^{25,26} which can be called main-path tunneling.²²

In a symmetric (not very asymmetric) SQUID besides the conventional main-path tunneling¹⁵ also multi-path tunneling is possible.²² In that case a density distribution under a barrier is not as in Fig. 2(b), but of the type as in Fig. 3(b). For a very asymmetric SQUID, considered in this paper, the both mechanisms result in the same tunneling exponents since the underbarrier channel shrinks to the line $y = y_0$ due to the mass difference.

We used a semiclassical approximation when there are many levels in the well. This approach sometimes is not appropriate in one-dimensional Josephson junctions where a barrier is weakly transparent but nevertheless there is only a few levels in the well (say, five).^{3,4,6-8} In a SQUID based on two such junctions the number of levels can be roughly estimated as 5×5 . In our case of a strongly asymmetric SQUID that number should be multiplied by the large parameter \sqrt{M} . Therefore the approximation of a large number of levels in the well is reasonable.

We propose two peculiarities of tunneling in an asymmetric SQUID which do not exist in a single junction and in a not very asymmetric SQUID.

One of them is temperature dependence of tunneling probability at low temperature. According to Eq. (31), the curves in Fig. 5 have a finite slope at low temperature. In one dimension the slope is zero due to the exponent $\exp(-const/T)$ instead of T in Eq. (31).

The second peculiarity is an unusual transition between thermally assisted tunneling and pure activation marked by dots in Fig. 5. At $2 < (1-j)\beta^2$ the transition is smooth but at $(1-j)\beta^2 < 2$ there are jumps of slopes in Fig. 5.

Low dissipation regime and parameters $M \simeq 35$ and $\beta \simeq 15$ correspond to reality in experiments with SQUIDs and fit the developed theory. It is more convenient in experiments to obtain a set of curves as in Fig. 6 since usually measurements are run at a fixed temperature. A dependence on temperature, as in Fig. 5, also can be obtained. This would provide an experimental check of the predicted dependences on temperature and current

VIII. CONCLUSION

Quantum tunneling in an asymmetric (with strongly different capacitances) SQUID is studied. Since capacitances play a role of masses one phase, related to a large mass, becomes "heavy" and remains always a constant in a tunneling process. Tunneling in an asymmetric SQUID becomes one-dimensional with a condition of optimization of tunneling probability with respect to a value of the "heavy" phase. An unusual temperature dependence of the tunneling probability is obtained. It has a finite slope at zero temperature and a transition between thermally assisted tunneling and pure activation can be not smooth depending on current through a SQUID.

Acknowledgments

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